Eigen value and eigen vector

Let, A be a square matrix of order n. The eqⁿ $|A-\lambda I|=0$ is called the characteristic eqⁿ of the matrix A. Clearly, $|A-\lambda I|=0$ is a polynomial in λ of degree n. The roots of this eqⁿ are called characteristic root or eigen value.

1) Find the eigen value's and eigen vector's of the matrix

$$A=\begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 0\\ 3 & 0 & 3\end{bmatrix}.$$

$\Rightarrow$ The characteristic eqⁿ is,

$$|A-\lambda I|=0$$

$$\text{or, } \begin{vmatrix}3-\lambda & 0 & 3\\ 0 & 3-\lambda & 0\\ 3 & 0 & 3-\lambda\end{vmatrix}=0$$

or, $(3-\lambda)(9+\lambda^2-6\lambda) \cancel{+} 3\{3(3-\lambda)\}=0$

or, $(3-\lambda)(\cancel{9}-6\lambda+\lambda^2-\cancel{9})=0$

or, $(3-\lambda)(\lambda^2-6\lambda)=0$

or, $3\lambda^2-18\lambda-\lambda^3+6\lambda^2=0$

or, $\lambda^3-9\lambda^2+18\lambda=0$

or, $\lambda^2-9\lambda+18=0$

or, $\lambda(\lambda-3)(\lambda-6)=0$

$\therefore \lambda=0, 3, 6$

$\therefore$ The eigen value's are 0, 3 and 6.

$\therefore$ The eigen vector corresponding to the

i) Let $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda=0$.

$$\therefore \begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 0\\ 3 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$$\therefore \left.\begin{aligned}3x_1+3x_3&=0\\ 3x_2&=0\\ 3x_1+3x_3&=0\end{aligned}\right\} \text{---(i)}$$

Solving the system of eqⁿ (i) we have,

$x_2=0$

$x_1 = -x_3$ $\qquad \therefore x_1=k$ and $x_3=-k$

$\dfrac{x_1}{1}=\dfrac{x_3}{-1}=k$

$\therefore$ The eigen vector is $X = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, k being Arbitrary.

ii) Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen ~~value~~ vector corresponding to the eigen value $\lambda = 3$.

$$\therefore \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \left.\begin{aligned} 3x_1 &= 0 \\ 3x_3 &= 0 \end{aligned}\right\} \text{—(ii)}$$

$\therefore x_1 = 0,\ x_3 = 0,\ x_2 = k'$ [k' being arbitrary realnumber]

$\therefore$ The eigen vector is $X = \begin{pmatrix} 0 \\ k' \\ 0 \end{pmatrix} = k'\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

iii) Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 6$.

$$\therefore \begin{bmatrix} -3 & 0 & 3 \\ 0 & -3 & 0 \\ 3 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \left.\begin{aligned} -3x_1 + 3x_3 &= 0 \\ -3x_2 &= 0 \\ 3x_1 - 3x_3 &= 0 \end{aligned}\right\} \text{—(iii)}$$

$\therefore x_2 = 0$

$x_1 = x_2$

$\frac{x_1}{1} = \frac{x_2}{1} = k''$ (k'' being arbitrary)

$x_1 = k''$, $x_2 = k''$

$\therefore$ The Eigen vector is $X = \begin{pmatrix} k'' \\ 0 \\ k'' \end{pmatrix} = k''\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

2) Find the eigen value's and eigen vectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$\Rightarrow$ The characteristic eqn is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$-\lambda(2-\lambda)(3-\lambda) + 2\times 2(2-\lambda) = 0$

or, $(2-\lambda)[\lambda(3-\lambda) + 4] = 0$

or, $(2-\lambda)[3\lambda - \lambda^2 + 4] = 0$

or, $\lambda = 2$ $\quad \lambda^2 - 3\lambda - 4 = 0$

or, $\lambda^2 - 4\lambda + \lambda - 4 = 0$

or, $\lambda(\lambda - 4) + 1(\lambda - 4) = 0$

$\lambda = 4, -1$

$\therefore$ The eigen value's are $-1, 2, 4$.

i) Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to $\lambda = -1$

$$\therefore \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore \left.\begin{aligned} x_1 + 2x_3 &= 0 \\ 3x_2 &= 0 \\ 2x_1 + 4x_3 &= 0 \end{aligned}\right\} \text{—(i)}$$

$\therefore x_2 = 0$

$x_1 = -2x_3$

$\frac{x_1}{1} = \frac{x_3}{-\frac{1}{2}} = k_1$

$x_1 = k_1, \quad x_3 = -\frac{k_1}{2}$ (k_1 being arbitrary)

or, $x_1 = 2k_1, \quad x_3 = -k_1$

$\therefore$ The eigen vector is

$$X = \begin{pmatrix} 2k_1 \\ 0 \\ -k_1 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

ii) Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 2$.

$$\therefore \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_2 = k_2$ (arbitrary)

$$\therefore \left.\begin{aligned} -2x_1 + 2x_3 &= 0 \\ 2x_1 + x_3 &= 0 \end{aligned}\right\} \text{—(ii)}$$

$x_1 = 0$

$x_3 = 0$

$\therefore$ The eigen vector is, $X = \begin{pmatrix} 0 \\ k_2 \\ 0 \end{pmatrix} = k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

iii) Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to $\lambda = 4$

$$\therefore \begin{pmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore \left.\begin{aligned} -4x_1 + 2x_3 &= 0 \\ -2x_2 &= 0 \\ 2x_1 - x_3 &= 0 \end{aligned}\right\} \text{—(iii)}$$

$x_2 = 0$

$2x_1 = x_3$

$\frac{x_1}{1} = \frac{x_3}{2} = k_3$ (k_3 being arbitrary)

$\therefore x_1 = k_3, \quad x_3 = 2k_3$

$\therefore$ The eigen vector is, $X = \begin{pmatrix} k_3 \\ 0 \\ 2k_3 \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Cayley Hamilton theorem

Statement: Every square matrix satisfies it's own characteristic equation.

1) verify Cayley - Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$

⇒ The charecteristic eq^n of the matrix,

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 1-\lambda & 2 \\ -2 & 3-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(3-\lambda) + 4 = 0$

or, $\lambda^2 - 4\lambda + 7 = 0$ —(i)

Now, we have,

$$A^2 - 4A + 7I$$

$= \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -3 & 8 \\ -8 & 5 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ -8 & 12 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -3-4+7 & 8-8 \\ -8+8 & 5-12+7 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ (verified)

2) Apply Cayley - Hamilton theorem to obtain A^{-1}, where $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$

⇒ The charecteristic eq^n of the matrix A,

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$

~~or, $(-\lambda)\lambda(4-\lambda) + 1\{3 - 2(1-\lambda)\} = 0$~~

~~or, $4\lambda^2 - \lambda^3 + 3 - 2 + 2\lambda = 0$~~

~~or, $\lambda^3 - 4\lambda^2 - 2\lambda - 1 = 0$~~

or, $-\lambda(1-\lambda)(4-\lambda) + 1\{3 + 2(1-\lambda)\} = 0$

~~or, $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$~~

or, $(-\lambda + \lambda^2)(4-\lambda) + 3 + 2 + 2\lambda = 0$

or, $-4\lambda + \lambda^2 + 4\lambda^2 - \lambda^3 + 5 + 2\lambda = 0$

or, $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$

∴ By Cayley-Hamilton theorem,

$$A^3 - 5A^2 + 6A - 5I = 0$$

or, $5I = A^3 - 5A^2 + 6A$

or, $5A^{-1} = A^2 - 5A + 6$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} - 5\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 5 \\ 15 & 5 & 0 \\ -10 & 5 & 20 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2+6 & 1 & 4-5 \\ 3-15 & 1-5+6 & 3 \\ -5+10 & 5-5 & 14-20+6 \end{pmatrix} = \begin{pmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix}$$

Some Results:-

i) Product of the eigen value's of a matrix is equal to it's determinant value

ii) Sum of the eigen value's of a matrix is equal to it's trace

iii) '0' is an eigen value of a singular matrix.

iv) The eigen value's of a diagonal matrix are it's diagonal elements.

v) The eigen value's of an upper triangular matrix are it's principle diagonal elements.

vi) The eigen value's of a non singular matrix are non zero.

vii) If λ is an eigen value of a non singular matrix A then λ^{-1} is an eigen value of A^{-1}.

viii) If λ is an eigen value of a non singular matrix A then λ^{-m} is an eigen value of A^{-m}; m being +ve integer.

ix) If λ is an eigen value of a matrix A then λ^{m} is an eigen value of A^{m}, m being +ve integer.

x) (Proof) The eigen values of a real ~~skew~~ symmetric matrix are all real.

xi) The eigen values of a real skew symmetric matrix are 0 or Purely imaginary.

xii) The eigen value's of a real orthogonal matrix are of unit modulus.
xiii) The eigen value's of A^T are the same as the eigen value's of A.
xiv) If λ be an eigen value of a real orthogonal matrix then λ^{-1} is also it's eigen value.

3) The trace and determinant of a (2×2) matrix are -2 and -35 respectively. Then the eigen value's of the matrix are —
i) 7, 5, ii) $-7, 5$, iii) $7, -5$, iv) $-7, -5$

⇒ We have,
the sum of eigen value's = the trace of the matrix
and the product of the eigen value's are it's determinant value.

4) The sum of the eigen value of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$
is i) 5, ii) 7, iii) 9, iv) 18
⇒ Sum of the eigen values = trace of the matrix
$= 1+5+1 = 7$.

5) If $A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$ then the value of A^{10} is i) $9^3 A$, ii) $3^3 A$,
iii) $3^9 \cdot A$, iv) none of these.
⇒ The charecteristic eq^n of the matrix A,
$$\begin{vmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$
or, $(2-\lambda)(1-\lambda) - 2 = 0$
or, $\lambda^2 - 3\lambda = 0$
∴ By Cayley-Hamilton theorem,
$A^2 - 3A = 0$
or, $A^2 = 3A$ —(i)
$\therefore A^{10} = (A^2)^5 = (3A)^5 = 3^5 A^5 = 3^5 (A^2)^2 \cdot A = 3^5 (3A)^2 A = 3^7 A^2 A$
$= 3^7 (3A) \cdot A = 3^8 A^2 = 3^8 \cdot 3A = 3^9 \cdot A$

6) Let, a and b be two +ve real numbers. Then the number of real eigen values of the matrix $\begin{pmatrix} a & 1 \\ 2 & b \end{pmatrix}$ — i) 0, ii) 1, iii) 2, iv) None
⇒ The charecteristic eq^n of the matrix,
$$\begin{vmatrix} a-\lambda & 1 \\ 2 & b-\lambda \end{vmatrix} = 0$$
or, $(a-\lambda)(b-\lambda) - 2 = 0$
or, $\lambda^2 - (a+b)\lambda + (ab-2) = 0$ —(i)
We have, $(a+b)^2 - 4(ab-2) = (a+b)^2 - 4ab + 8 = (a-b)^2 + 8 > 0$

∴ All the roots of the eq^n (1) are real and hence all the eigen value's of the given matrix are real.

Rank of a matrix

Definition :- A positive integer r is said to be the rank of a non zero matrix $A_{m\times n}$ if :- i) there is atleast one $r\times r$ submatrix of A whose determinant is not equal to zero. and
ii) the determinant of every $(r+1)\times(r+1)$ submatrix of A is zero.

[Note :- rank of the matrix $A_{m\times n}$ is denoted by $r(A)$ and $0 < r(A) \le \min(m,n)$

For example, the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ is 2. Since,
$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0$

1) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

⇒ we have,
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1(15-16) - 2(10-12) + 3(8-9) = 0$$

$\therefore r(A) \neq 3$

Since, $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$

$\therefore r(A) = 2$

2) Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \end{bmatrix}$

⇒ Since,
$$\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -1(-1) - 3(1) = -2 \neq 0$$

$\therefore r(A) = 3$

Row reduced Echelon form :-

3) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ to it's row reduced Echelon form and hence determine it's rank.

⇒
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3' = R_3 - 3R_1 \\ R_4' = R_4 - R_1}]{R_2' = R_2 - 4R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4' = R_4 + \frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row reduced echelon form of the given matrix.

Since, the number of non zero row's is three.
Therefore, the rank of the matrix A is 3.

4) Reduce the matrix A to row reduced echelon form and hence find it's rank where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ -1 & 1 & -2 & 0 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ -1 & 1 & -2 & 0 \end{bmatrix} \xleftrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ -1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[R_4' = R_4 + R_1]{R_3' = R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4' = R_4 - R_2]{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xleftrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row reduced echelon form of the given matrix.

Since, the number of non zero row's is three

$\therefore \rho(A) = 3$.

Normal form :-

5) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ to it's normal form and hence find it's rank.

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow[C_3' = C_3 - 3C_1]{C_2' = C_2 - 2C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2' = -R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{R_3' = R_3 + 3R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_3' = C_3 - 2C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$. $\rightarrow \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

6) Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ to the normal form and find it's rank.

$$\Rightarrow A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3' = R_3 - 3R_1 \\ R_4' = R_4 - 6R_1}]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \xrightarrow[\substack{C_3' = C_3 + 2C_1 \\ C_4' = C_4' + 4C_1}]{C_2' = C_2 + C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\xrightarrow{R_2' = R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \xrightarrow[R_4' = R_4 - 9R_2]{R_3' = R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$\xrightarrow[C_4' = C_4 + 3C_2]{C_3' = C_3 + 6C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$\xrightarrow{R_4' = R_4 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_4' = C_4 - \frac{2}{3}C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3' = \frac{1}{33}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore r(A) = 3$

Congruence operation:-

7) Obtain the normal form under congruence and find the rank and signature of the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$\Rightarrow$

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{bmatrix} \xrightarrow{R_2' = R_2 - 2R_1} \begin{bmatrix} 2 & 4 & 3 \\ 0 & -2 & -3 \\ 3 & 3 & 1 \end{bmatrix} \xrightarrow{C_2' = C_2 - 2C_1} \begin{bmatrix} 2 & 0 & 3 \\ 0 & -2 & -3 \\ 3 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 0 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -\frac{7}{2} \end{pmatrix} \xrightarrow{C_3' = C_3 - \frac{3}{2}C_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -3 & -\frac{7}{2} \end{pmatrix} \xrightarrow{R_3' = R_3 - \frac{3}{2}R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{C_3' = C_3 - \frac{3}{2}C_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1' = \frac{1}{\sqrt{2}}R_1} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1' = \frac{1}{\sqrt{2}}C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2' = \frac{1}{\sqrt{2}}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2' = C_2 \times \frac{1}{\sqrt{2}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

~~$\therefore r(A) = 3$~~

$\therefore$ rank of the matrix, $r = 3$, Here $m = 2$ (number of positive ones in the principle diagonal)

$\therefore$ Signature $= 2m - r = 4 - 3 = 1$

System of eq^n

Consider, the system of equations
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}\right\} \text{---(i)}$$

The system (i) can be written as $AX = b$ — (ii), where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Let, $$A_G = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$ (Augmented / ~~Partion~~ Partition matrix)

The system of eq^n (i) is said to be consistent if $r(A) = r(A_G)$

The system of eq^n (i) is said to be inconsistent if $r(A) \neq r(A_G)$

Nature of solution:– i) If $r(A) = r(A_G) = n$ then the system of eq^n (i) has unique solution.

ii) If $r(A) = r(A_G) < n$ then the system of eq^n (i) has infinite number of solutions.

iii) If $r(A) \neq r(A_G)$ then the system of eq^n (i) has no solution.

for example i) consider the system of eq^n $2x_1 - x_2 = 2$
$4x_1 - 2x_2 = 3$

Here, $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ and $A_G = \begin{pmatrix} 2 & -1 & 2 \\ 4 & -2 & 3 \end{pmatrix}$

$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ $-3+4 = 1 \neq 0$

We have, $r(A) = 1$ and $r(A_G) = 2$

Since, $r(A) \neq r(A_G)$

$\therefore$ the system has no solution.

ii) Consider the system of eq^n $x_1 + 2x_2 = 3$
$2x_1 + 4x_2 = 6$

Here, $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $A_G = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 4-4 =$
$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12$
$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6-$

We have, $r(A) = 1$ and $r(A_G) = 1$

Since, $r(A) = r(A_G) = 1 < 2$ (number of variables)

$\therefore$ The system has infinite number of solution.

iii) Consider the system of eq^n $2x_1 - x_2 = 4$
$x_1 + x_2 = 6$

Here, $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $A_G = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 1 & 6 \end{pmatrix}$

$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$
$= 2+1 = 3 \neq 0$

We have, $r(A) = 2$ and $r(A_G) = 2$

Since, $r(A) = r(A_G) = 2 =$ number of variables.

$\therefore$ The system has unique solution

$\therefore$ The solution $= \left(\frac{10}{3}, \frac{8}{3}\right)$

1) Investigate for what values of λ and μ, the following eq^{n}s

$x+y+z=6$
$x+2y+3z=10$
$x+2y+\lambda z=\mu$

have — i) unique solution, ii) infinite number of solution and iii) No solution?

$\Rightarrow$ Here, $A=\begin{pmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{pmatrix}$ and $A_G=\begin{pmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{pmatrix}$

We have,

$$A_G=\begin{pmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{pmatrix}\xrightarrow[R_3'=R_3-R_1]{R_2'=R_2-R_1}\begin{pmatrix}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{pmatrix}\xrightarrow{R_3'=R_3-R_2}\begin{pmatrix}1&1&1&6\\0&1&2&0\\0&0&\lambda-3&\mu-10\end{pmatrix}$$

i) If $\lambda\neq 3$ then $r(A)=3$, $r(A_G)=3$
Since, $r(A)=r(A_G)=3=$ number of variables
$\therefore$ The given system has unique solution for $\lambda\neq 3$, whatever μ may be.

ii) If $\lambda=3$ and $\mu=10$ then $r(A)=2$ and $r(A_G)=2$
Since, $r(A)=r(A_G)=2<3$ (number of variables)
$\therefore$ The given system has infinite number of solution for $\lambda=3$ and $\mu=10$

iii) If $\lambda=3$ and $\mu\neq 10$ then $r(A)=2$ and $r(A_G)=3$
Since, $r(A)\neq r(A_G)$, the given system has no solution in this case.

2) Find for what values of a and b, the system of eq^{n}s

$x+y+z=1$
$x+2y-z=b$
$5x+7y+az=b^2$

has — i) a unique solution, ii) No solution, iii) Infinite number of solutions?

$\Rightarrow$ Here, $A=\begin{pmatrix}1&1&1\\1&2&-1\\5&7&a\end{pmatrix}$ and $A_G=\begin{pmatrix}1&1&1&1\\1&2&-1&b\\5&7&a&b^2\end{pmatrix}$

We have,

$$A_G=\begin{pmatrix}1&1&1&1\\1&2&-1&b\\5&7&a&b^2\end{pmatrix}\xrightarrow[R_3'=R_3-5R_1]{R_2'=R_2-R_1}\begin{pmatrix}1&1&1&1\\0&1&-2&b-1\\0&2&a-5&b^2-5\end{pmatrix}\xrightarrow{R_3'=R_3-2R_1}\begin{pmatrix}1&1&1&1\\0&1&-2&b-1\\0&2&a-1&b^2-2b-3\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1&1&1&1\\0&1&-2&b-1\\0&2&a-1&(b+1)(b-3)\end{pmatrix}$$

i) If $a\neq 1$, then $r(A)=3$ and $r(A_G)=3$
Since, $r(A)=r(A_G)=3=$ number of variables
$\therefore$ The given system of eq^{n}s has a unique solution in this case.

ii) If $a=1$ and $b\neq -1, 3$ then $r(A)=2$ and $r(A_G)=3$
Since, $r(A)\neq r(A_G)$, the given system has no solution in this case.

iii) If $a=1$, $b=-1$ and 3 then $r(A)=2$ and $r(A_G)=2$
Since, $r(A)=r(A_G)=2<3$ (number of variables)
$\therefore$ The given system has infinite number of solutions.

Cramer's Rule: Consider the system of eq^ns

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n &= b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n &= b_2 \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n &= b_n \end{aligned}\right\} \text{--- (i)}$$

i.e. $AX = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_n \end{pmatrix}$$

and

$$A_a = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

If $|A| \neq 0$ then $r(A) = n$ consequently, $r(A_a) = n$.

$\therefore$ The system of eq^ns (i) has unique solution, if $|A| \neq 0$.

Let,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ - & - & - & - \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix} \; \dots\dots, \quad \Delta_n = \begin{vmatrix} a_{11} & a_{21} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix}$$

Then, the solution of the system (i) is given by

$$\frac{x_1}{\Delta_1} = \frac{x_2}{\Delta_2} = \frac{x_3}{\Delta_3} = \dots = \frac{x_n}{\Delta_n} = \frac{1}{\Delta}$$

3) Solve the following system of eq^ns by Cramer's Rule

$$x+2y+3z = 6$$
$$2x+4y+z = 7$$
$$3x+2y+9z = 14$$

$\Rightarrow$ Here,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = 1(36-2)-2(18-3)+3(4-12)$$
$$= 34-30-24 = -20 \neq 0$$

$\therefore$ Cramer's rule can be applied.

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = 6(36-2) - 2(63-14)+3(14-56)$$
$$= -20$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = 1(63-14)-6(18-3)+3(28-21)$$
$$= 49 - 90 + 21 = -90+70$$
$$= -20$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = 1(56-14)-2(28-21)+6(4-12)$$
$$= 42 - 14 - 48 = 42-14-6$$

∴ The solution is given by $\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$

i.e. $\frac{x}{-20} = \frac{y}{-20} = \frac{z}{-20} = \frac{1}{-20}$

∴ $x = 1,\ y = 1,\ z = 1$

4) Solve the following system of eqn by Cramer's Rule

$x + 2y + 3z = 4$
$2x + 3y + 4z = 5$
$3x + 4y + 5z = 7$

5) Find for what values of a and b, the system of eqn's

$x + 4y + 2z = 1$
$2x + 7y + 5z = 2b$
$4x + ay + 10z = 2b + 1$

has — i) unique solution, ii) infinite number of solutions, and iii) No solution

6) Determine the value of k, so that the following system of eqn's has — i) no solution, ii) infinite number of solutions and iii) unique solutions

$x + y - z = 1$
$2x + 3y + kz = 3$
$x + ky + 3z = 2$

7) Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{pmatrix}$. Using this, solve the following system of eqn's.

$x + y + 2z = 4$
$2x - y + 3z = 9$
$3x - y - z = 2$

⇒ $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{pmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 1(1+3) - 1(-2-9) + 2(-2+3)$
$= 4 + 11 + 2 = 17$

The given system can be written as $AX = b$ — (i), where $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 9 \\ 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore X = A^{-1} b$ — (ii)

Now, $\operatorname{adj}(A) = \begin{bmatrix} (1+3) & -(-2-9) & (-2+3) \\ -(-1+2) & (-1-6) & -(-1-3) \\ (3+2) & -(3-4) & (-1-2) \end{bmatrix}^T = \begin{bmatrix} 4 & 11 & 1 \\ -1 & -7 & 4 \\ 5 & 1 & -3 \end{bmatrix}^T$

$= \begin{pmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{pmatrix}$

$\therefore A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{1}{17} \begin{pmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{pmatrix}$

$\therefore X = A^{-1} b = \frac{1}{17} \begin{pmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 9 \\ 2 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 16 - 9 + 10 \\ 44 - 63 + 2 \\ 4 + 36 - 6 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 17 \\ -17 \\ 34 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore x = 1, \; y = -1, \; z = 2$

8) Determine the rank of the following matrix for different values of λ, $A = \begin{bmatrix} \lambda & 1 & 0 \\ 3 & \lambda-2 & 1 \\ 3(\lambda+1) & 0 & \lambda+1 \end{bmatrix}$.

9) The rank of the matrix $\begin{pmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{pmatrix}$ is i) 4 ii) 3 iii) 2 iv) 1

10) The rank of the matrix $A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$ is i) 4 ii) 3 iii) 2 iv) 1

11) Let, A be a non singular matrix of order 3 then the rank of the matrix A^3 is i) 1, ii) 2, iii) 3, iv) 9

$\Rightarrow |A| \neq 0, \; |A^2| \neq 0, \; |A^3| \neq 0$

12) The rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ is i) 1, ii) 2, iii) 3, iv) 4

Linear transformation

Linear Combination:- Let, V be a vector space over the field F. If $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ than any vector α is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ if there exists scalers $c_1, c_2, \dots, c_n \in F$ such that
$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$

1) Express $(-1,2,4)$ as a linear combination of the vectors $(-1,2,0)$, $(0,-1,1)$ and $(3,-4,2)$.

$\Rightarrow$ Let,

$$(-1,2,4) = c_1(-1,2,0) + c_2(0,-1,1) + c_3(3,-4,2) \quad \text{---(i)}$$

$$\therefore \left.\begin{aligned} -c_1 + 0\cdot c_2 + 3c_3 &= -1 \\ 2c_1 - c_2 - 4c_3 &= 2 \\ 0\cdot c_1 + c_2 + 2c_3 &= 4 \end{aligned}\right\} \text{---(ii)}$$

Solving the system (ii) we have, $c_1 = 4$, $c_2 = 2$ and $c_3 = 1$

$\therefore$ from (i), $(-1,2,4) = 4(-1,2,0) + 2(0,-1,1) + 1(3,-4,2)$.

Linearly dependent vector:- Let, V be a vector space over a field F. The vectors $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ are said to be linearly dependent (L.D) if there exists scalers $c_1, c_2, \dots, c_n \in F$, not all zero, such that $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \theta$, θ being null vector of V.

Linearly independent vector (L.I):- Let, V be a vector space over a field F. The vectors $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ are said to be linearly independent (L.I) if $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \theta$ holds only when all the scalers $c_1, c_2, \dots, c_n$ are zero.

2) Show that the vectors $(-1,2,1)$, $(3,0,-1)$ and $(-5,4,3)$ are linearly dependent in R^3.

$\Rightarrow$ Let, $c_1(-1,2,1) + c_2(3,0,-1) + c_3(-5,4,3) = (0,0,0)$ ---(i)

from (i) we have,

$$\left.\begin{aligned} -c_1 + 3c_2 - 5c_3 &= 0 \\ 2c_1 + 0\cdot c_2 + 4c_3 &= 0 \\ c_1 - c_2 + 3c_3 &= 0 \end{aligned}\right\} \text{---(ii)}$$

Solving the system (ii) we have, $c_1 = -2$, $c_2 = 1$, $c_3 = 1$

$\therefore$ from (i), the given vectors are linearly dependent.

Alternative method, we have,

$$\begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{vmatrix} = 0$$

$\therefore$ The given vectors are linearly dependent.

3) Show that the vectors $(2,-3,1)$, $(3,-1,5)$ and $(1,-4,3)$ of R^3 are L.I.

$\Rightarrow$ We have,
$$\begin{vmatrix} 2 & -3 & 1 \\ 3 & -1 & 5 \\ 1 & -4 & 3 \end{vmatrix} = 2(-3+20)+3(9-5)+1(-12+1) = 34+12-11 = 35 \neq 0$$

$\therefore$ The given vectors are L.I.

(2) **Basis** :- Let, V be a vector space over the field F. The set of vectors $B=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is said to be basis of V if — i) the vectors of B are ~~are~~ L.I. and ii) if α be any vector of V then α can be expressed as a linear combination of the vectors of B, i.e. there exists scalers $c_1, c_2, \dots, c_n \in F$ such that $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$

For example, i) $B=\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of R^3. This basis of R^3 is called standard basis.

ii) $B=\{(1,0),(0,1)\}$ is a standard basis of R^2. or $V_2(R)$

(1) **Dimension** :- The number of vectors in the basis of a finite dimensional vector space, V over the field F is called the dimension of the vector space and it's denoted by dimV.

For example, i) Dimension of R^3 or $V_3(R)$ is 3.

ii) Dimension of $V_2(R)$ is 2.

4) Show that the vectors $(1,2,1)$, $(2,1,0)$, $(1,-1,2)$ forms a basis of $V_3(R)$.

$\Rightarrow$ We have,
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 1(2) - 2(4) + 1(-2-1) = 2-8-3 = -9 \neq 0$$

$\therefore$ The given vectors are L.I.

Let, (a,b,c) be any vector of $V_3(R)$.

Consider,
$$(a,b,c) = c_1(1,2,1)+c_2(2,1,0)+c_3(1,-1,2) \quad \text{—— (i)}$$

from (i),
$$\left.\begin{aligned} c_1+2c_2+c_3 &= a \\ 2c_1+c_2-c_3 &= b \\ c_1+0.c_2+2c_3 &= c \end{aligned}\right\} \text{—— (ii)}$$

the determinant of the coefficient matrix of the system of eq^n (ii) is
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = -9 \neq 0$$

$\therefore$ The system (ii) has unique solution.

$\therefore$ The vector (a,b,c) can be expressed uniquely as a linear combination of the given vectors.

$\therefore$ The given vectors forms a ~~basis~~ basis of $V_3(R)$

Subspace:- Let, V be a vector space over the field F and W be a non empty subset of V. W is said to be a subspace of V if W itself a vector space over the same field F.

Theorem:- ~~Statem~~

Statement:- A non empty subset W of a vector space V over the field F is a subspace of V if and only if i) $\alpha, \beta \in W \Rightarrow \alpha+\beta \in W$

ii) $c \in F$ and $\alpha \in W \Rightarrow c\alpha \in W$

5) Let, $W = \{(x,y,z) \in R^3 : 3x - y + z = 0\}$. Show that W is a subspace of R^3.

$\Rightarrow$ Clearly, W is non empty subset of R^3, since $(0,0,0) \in W$.

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in W$

$\therefore 3x_1 - y_1 + z_1 = 0$ —(i)

$3x_2 - y_2 + z_2 = 0$ —(ii)

Now, $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

We have, $3(x_1+x_2) - (y_1+y_2) + (z_1+z_2)$

$= (3x_1 - y_1 + z_1) + (3x_2 - y_2 + z_2)$

$= 0 + 0$ [by (i) and (ii)]

$= 0$

$\therefore \alpha + \beta \in W$

Let, c be any scaler.

$\therefore c\alpha = (cx_1, cy_1, cz_1)$

Now we have, $3cx_1 - cy_1 + cz_1 = c(3x_1 - y_1 + z_1) = c(0)$ [by (i)] $= 0$

$\therefore c\alpha \in W$

$\therefore W$ is a subspace of R^3.

6) Show that $W = \{(x,y,z) \in R^3 : x+y+z=0\}$ is a subspace of R^3. Find a basis of W.

$\Rightarrow$ Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in W$

$\therefore x_1 + y_1 + z_1 = 0$ —(i)

$x_2 + y_2 + z_2 = 0$ —(ii)

We have, $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

Now, $(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = 0 + 0$ [by (i) and (ii)] $= 0$

$\therefore \alpha + \beta \in W$

also, $c\alpha = (cx_1, cy_1, cz_1)$

$cx_1 + cy_1 + cz_1 = c(x_1+y_1+z_1) = c \cdot 0$ [by (i)] $= 0$

$\therefore c\alpha \in W$

$\therefore W$ is a subspace of R^3.

Let, $\gamma = (x, y, z)$ be any vector of W.

$\therefore x+y+z=0$

$\Rightarrow z = -x-y$

$\therefore \gamma = (x, y, z)$

$= (x, y, -x-y)$

$= x(1,0,-1) + y(0,1,-1)$

Let; $B = \{(1,0,-1), (0,1,-1)\}$

~~Consider~~, $\therefore$ Any vector of W can be expressed as a linear combination of the vectors of B.

Consider, $c_1(1,0,-1) + c_2(0,1,-1) = (0,0,0)$ —— (i)

from (i),
$$\left.\begin{aligned} c_1 + 0.c_2 &= 0 \\ 0.c_1 + c_2 &= 0 \\ -c_1 - c_2 &= 0 \end{aligned}\right\} \text{—— (ii)}$$

The system (ii) has only trivial solution which is $c_1 = 0$ and $c_2 = 0$.

$\therefore$ from (i), The vectors of B are linearly independent.

$\therefore B$ is a basis of W.

[Note:— dimension of $W = 2$.]

7) Find a basis of $W = \{(x, y, z) \in \mathbb{R}^3 : 3x - y + z = 0\}$

$\Rightarrow$ Let, $\gamma = (x, y, z)$ be any vector of W.

$\therefore 3x - y + z = 0$

$\Rightarrow y = 3x + z$

$\therefore \gamma = (x, y, z) = (x, 3x+z, z) = x(1,3,0) + z(0,1,1)$

Let, $B = \{(1,3,0), (0,1,1)\}$

$\therefore$ Any vector of W can be expressed as a linear combination of the vectors of B.

Let us consider, $c_1(1,3,0) + c_2(0,1,1) = (0,0,0)$ —— (i)

$\therefore$ from (i),
$$\left.\begin{aligned} c_1 + 0.c_2 &= 0 \\ 3c_1 + c_2 &= 0 \\ 0.c_1 + c_2 &= 0 \end{aligned}\right\} \text{—— (ii)}$$

The system (ii) has only trivial solution which is $c_1 = 0$ and $c_2 = 0$.

$\therefore$ from (i), the vectors of B are linearly independent.

$\therefore B$ is a basis of W.

Linear transformation (L.T.)

Definition:- Let, V and W be vector spaces over the same field F. A mapping $T: V \to W$ is said to be a linear transformation or linear mapping if the following conditions are satisfied, $\forall$ vector $\alpha, \beta \in V$ and $\forall$ all scales $c \in F$: i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$ (Additive Property)
and ii) $T(c\alpha) = cT(\alpha)$ (homogenious property)

Example, Consider the mapping $T: \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x,y,z) = z$.
Let, $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2) \in \mathbb{R}^3$

$\therefore T(\alpha) = T(x_1, y_1, z_1) = z_1$

$T(\beta) = T(x_2, y_2, z_2) = z_2$

Now, $\alpha+\beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

and $c\alpha = (cx_1, cy_1, cz_1)$

$\therefore T(\alpha+\beta) = T(x_1+x_2, y_1+y_2, z_1+z_2) = z_1+z_2 = T(\alpha)+T(\beta)$

and $T(c\alpha) = T(cx_1, cy_1, cz_1) = cz_1 = cT(\alpha)$

$\therefore T$ is a linear transformation.

Consider the mapping $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x,y) = xy$.
Let, $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2) \in \mathbb{R}^2$

$\therefore T(\alpha) = T(x_1, y_1) = x_1y_1$ and $T(\beta) = T(x_2, y_2) = x_2y_2$

Now, $\alpha+\beta = (x_1+x_2, y_1+y_2)$

$\therefore T(\alpha+\beta) = (x_1+x_2)(y_1+y_2) = x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1 \neq T(\alpha)+T(\beta)$

$\therefore T$ is not a linear transformation.

Some properties :-
Let, $T: V \to W$ be a linear transformation. then where θ and θ' are the null vectors of V and W respectively.

i) $T(\theta) = \theta'$

ii) $T(\alpha-\beta) = T(\alpha) - T(\beta)$

iii) $T(-\alpha) = -T(\alpha)$

Matrix of a linear transformation:- Let, V and W be vector spaces over same field F and $T: V \to W$ be a linear transformation.
Let, $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be the bases of the vector space V and W respectively.

Let, $T(\alpha_1) = a_{11}\beta_1 + a_{12}\beta_2 + \ldots + a_{1m}\beta_m$

$T(\alpha_2) = a_{21}\beta_1 + a_{22}\beta_2 + \ldots + a_{2m}\beta_m$

$\cdots$

$T(\alpha_n) = a_{n1}\beta_1 + a_{n2}\beta_2 + \ldots + a_{nm}\beta_m$

The matrix of the linear transformation T is denoted by $m(T)$ and defined by

$$m(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}^T$$

1) The matrix representation of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$ relative to the basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$ is —

i) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, ii) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, iii) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, iv) none of these.

⇒ We have,

$T(1,0) = (1,1) = 1(1,0) + 1(0,1)$

$T(0,1) = (1,-1) = 1(1,0) + (-1)(0,1)$

$\therefore m(T) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

2) Let, $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x+y, y+z)$. Find the matrix of the linear transformation with respect to the standard bases.

⇒ The standard basis of $\mathbb{R}^3$ is $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and that of $\mathbb{R}^2$ is $B' = \{(1,0),(0,1)\}$.

We have, $T(1,0,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1,0) = (1,1) = 1(1,0) + 1(0,1)$

$T(0,0,1) = (0,1) = 0(1,0) + 1(0,1)$

$\therefore$ The matrix of the given linear transformation is $m(T) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}^T$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

3) Find out the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x+3y+2z, 4x+9z)$ with respect to the standard Basis.

⇒ The standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ are respectively $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B' = \{(1,0),(0,1)\}$

$\therefore$ We have, $T(1,0,0) = (1,4) = 1(1,0) + 4(0,1)$

$T(0,1,0) = (3,0) = 3(1,0) + 0(0,1)$

$T(0,0,1) = (2,9) = 2(1,0) + 9(0,1)$

$\therefore$ The matrix of the given linear transformation is $m(T) = \begin{pmatrix} 1 & 4 \\ 3 & 0 \\ 2 & 9 \end{pmatrix}^T$

$= \begin{pmatrix} 1 & 3 & 2 \\ 4 & 0 & 9 \end{pmatrix}$

4) Let, $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y,z) = (3x+2y-4z, x-5y+3z)$. Find the matrix representation of T relative to the bases $\{(1,1,1),(1,1,0),(1,0,0)\}$ and $\{(1,3),(2,5)\}$ respectively of $\mathbb{R}^3$ and $\mathbb{R}^2$.

⇒ We have, $T(1,1,1) = (1,-1) = -7(1,3) + 4(2,5)$

$T(1,1,0) = (5,-4) = -33(1,3) + 19(2,5)$

$T(1,0,0) = (3,1) = -13(1,3) + 8(2,5)$

$\therefore$ The matrix, $m(T) = \begin{pmatrix} -7 & 4 \\ -33 & 19 \\ -13 & 8 \end{pmatrix}^T$

$= \begin{pmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{pmatrix}$

$(1,-1) = a_1(1,3) + a_2(2,5)$

$a_1 + 2a_2 = 1$

$3a_1 + 5a_2 = -1$

$3a_{21} + 6a_{22} = 15$

$3a_{21} + 5a_{22} = -4$

$a_{22} = 19$

$\therefore a_{21} = -4 - 45$

$3a_{31} + 6a_{32} = 9$

$3a_{31} + 5a_{32} = 1$

$a_{32} = 8$

$a_{31} = 3 - 16$

$= -13$

5) In $T: \mathbb{R}^2 \to \mathbb{R}^2$, T maps $(1,1)$ to $(2,-3)$ and $(1,-1)$ to $(4,7)$. Find the matrix of T relative to the standard basis.

$\Rightarrow$ given that, $T(1,1) = (2,-3)$ and $T(1,-1) = (4,7)$

and standard basis of $\mathbb{R}^2 = \{(1,0),(0,1)\}$.

we have, $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$

$\therefore T(1,0) = T\{\frac{1}{2}(1,1) + \frac{1}{2}(1,-1)\} = T\{\frac{1}{2}(1,1)\} + T\{\frac{1}{2}(1,-1)\}$

$= \frac{1}{2}T(1,1) + \frac{1}{2}T(1,-1) = \frac{1}{2}(2,-3) + \frac{1}{2}(4,7) =$

$= (3,2) = 3(1,0) + 2(0,1)$

Again, $(0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

$\therefore T(0,1) = T\{\frac{1}{2}(1,1) - \frac{1}{2}(1,-1)\} = T\{\frac{1}{2}(1,1)\} - T\{\frac{1}{2}(1,-1)\} = \frac{1}{2}T(1,1) - \frac{1}{2}T(1,-1)$

$= \frac{1}{2}(2,-3) - \frac{1}{2}(4,7) = (-1,-5) = -1(1,0) - 5(0,1)$

$\therefore m(T) = \begin{pmatrix} 3 & 2 \\ -1 & -5 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix}$

6) If $T: \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation, then show that $T(x,y) = mx + ny$ for some $m, n \in \mathbb{R}$.

$\Rightarrow$ Let, $B = \{u, v\}$ be the basis of $\mathbb{R}^2$.

$T(x,y) = T(au+bv) = T(au) + T(bv) = aT(u) + bT(v) = ax + by$ $\left[\begin{array}{l} x = T(u) \text{ and} \\ y = T(v) \end{array}\right]$.

7) For what value of k, the set $\{(k,1,1),(1,k,1),(1,1,k)\}$ is linearly independent.

$\Rightarrow$ We have, $\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = k(k^2-1) - 1(k-1) + 1(1-k) = k^3 - 3k + 2$

$= k^3 - k - k + 1 + 1 - k =$

$= (k-1)(k^2+k-2) = (k-1)(k+2)(k-1)$

$= (k-1)^2(k+2)$

$\therefore$ The given vectors will be linearly independent if $k \neq 1$ and $k \neq -2$.

Matrix Polynomial

Definition:- Let, $F(x) = A_0 + A_1x + A_2x^2 + \dots + A_nx^n$, where $A_0, A_1, \dots, A_n$ are matrices of same order. $F(x)$ is called matrix polynomial in x of degree 'n'.

Example:- $F(x) = \begin{bmatrix} x^2+2x & 5x \\ x^3+3 & 5 \end{bmatrix} = \begin{bmatrix} 0+2x+x^2+0.x^3 & 0+5x+0.x^2+0.x^3 \\ 3+0.x+0.x^2+x^3 & 5+0.x+0.x^2+0.x^3 \end{bmatrix}$

$\therefore F(x) = \begin{bmatrix} 0 & 0 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}x^2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}x^3 = A_0 + A_1x + A_2x^2 + A_3x^3$

Degree of the matrix polynomial is 3.

Cayley-Hamilton theorem:- ☆

Statement:- every square matrix satisfies it's own characteristic equation.

Proof:- Let, A be a square matrix of order n. Let the characteristic eq^n of A be, $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$ —— (i)

We are to show that $a_0I + a_1A + a_2A^2 + \dots + a_nA^n = 0$

Clearly, $|A - xI| = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ —— (ii)

Now, each elements of $adj(A-xI)$ are simple polynomial in x and among these polynomials the highest degree of the polynomial is $(n-1)$.

$\therefore adj(A-xI)$ can be expressed as a matrix polynomial of degree $(n-1)$.

Let, $adj(A-xI) = B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1}$ —— (iii)

where, $B_0, B_1, \dots, B_{n-1}$ are each square matrix of order n.

Again we have, $[\because A\, adj(A) = |A|I]$

$$(A - xI)\, Adj(A - xI) = |A - xI|.I = (a_0 + a_1x + a_2x^2 + \dots + a_nx^n)I$$

$$\Rightarrow (A - xI)(B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1}) = a_0I + a_1xI + a_2Ix^2 + \dots + a_nIx^n \quad \text{—— (iv)}$$

$\therefore (A - xI)(B_0 + B_1x + B_2x^2$ from (iv) we have,

Comparing the like powers of x

$$\left.\begin{aligned} AB_0 &= a_0I \\ AB_1 - B_0 &= a_1I \\ AB_2 - B_1 &= a_2I \\ AB_3 - B_2 &= a_3I \\ AB_4 - B_3 &= a_4I \\ &\;\;\vdots \\ AB_{n-1} - B_{n-2} &= a_{n-1}I \\ - B_{n-1} &= a_nI \end{aligned}\right\} \text{—— (v)}$$

Pre-multiplying each eq^{n}s of (v) by $I, A, A^2, A^3, \dots, A^n$ respectively and adding columnwise we have, $a_0I + a_1A + a_2A^2 + a_3A^3 + \dots + a_nA^n = 0$

Hence, the theorem is proved.

1) Let, $A=\begin{pmatrix}1 & 3\\ 3 & -7\end{pmatrix}$ verify Cayley-Hamilton theorem for the matrix A.

$\Rightarrow$ We have, the characteristic eqⁿ of A,

$$|A-xI|=0$$

$$\begin{vmatrix}1-x & 3\\ 3 & -7-x\end{vmatrix}=0$$

$$\Rightarrow (1-x)(-7-x)-9=0$$

$$\Rightarrow x^2+6x-16=0$$

Now we have, $A^2+6A-16I_2$

$$=\begin{pmatrix}1 & 3\\ 3 & -7\end{pmatrix}\begin{pmatrix}1 & 3\\ 3 & -7\end{pmatrix}+\begin{pmatrix}6 & 18\\ 18 & -42\end{pmatrix}-\begin{pmatrix}16 & 0\\ 0 & 16\end{pmatrix}$$

$$=\begin{pmatrix}10 & -18\\ -18 & 58\end{pmatrix}+\begin{pmatrix}6 & 18\\ 18 & -42\end{pmatrix}-\begin{pmatrix}16 & 0\\ 0 & 16\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$=0$ (verified)

2) The eigen values of the matrix $A=\begin{pmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{pmatrix}$ are — i) 1, 1, 1 ii) 1, 1, 2 iii) 1, 4, 4 iv) 1, 2, 4

$\Rightarrow$ We know that, sum of eigen values of a matrix = the trace of the matrix

$\therefore$ The eigen values are, 1, 4, 4.

3) Let, $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ then Prove that $v_1=e_1+e_2$ and $v_2=e_1-e_2$ are the eigen vectors with eigen values 1 and -1 respectively, where e_1 and e_2 having their usual meanings.

$\Rightarrow$ Let, $X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ be the eigen vector corresponding to the eigen value 1.

$$\therefore \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore \left.\begin{aligned}-x_1+x_2&=0\\ x_1-x_2&=0\end{aligned}\right\} \text{— (i)}$$

Solving (i)

$\therefore x_1=x_2=k$

$\therefore x=\begin{pmatrix}k\\ k\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$ (Choosing $k=1$)

$$=1\begin{pmatrix}1\\ 0\end{pmatrix}+1\begin{pmatrix}0\\ 1\end{pmatrix}=e_1+e_2$$

Let, $y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ be the eigen vector corresponding to the eigen value -1.

$$\therefore \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore \left.\begin{aligned}y_1+y_2&=0\\ y_1+y_2&=0\end{aligned}\right\} \text{— (ii)}$$

Solving (ii), $\frac{y_1}{1}=\frac{y_2}{-1}=k$

$y_1=1$, $y_2=-1$ [Choosing $k=1$]

$$\therefore y=\begin{pmatrix}1\\ -1\end{pmatrix}=1\begin{pmatrix}1\\ 0\end{pmatrix}-1\begin{pmatrix}0\\ 1\end{pmatrix}=e_1-e_2$$

4) A matrix M has eigen values 1 and 4. With corresponding eigen vectors $(1,-1)^t$ and $(2,1)^t$ respectively. Find the matrix M.

⟹ Let, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ $\quad \therefore a-1-b=0$ ——(i)

$c-d+1=0$ ——(ii)

Again, $\begin{pmatrix} a-4 & b \\ c & d-4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$ $\quad 2a-8+b=0$ ——(iii)

$2c+d-4=0$ ——(iv)

$a-b-1=0$

$2a+b-8=0$

$\frac{a}{8+1} = \frac{b}{-2+8} = \frac{1}{1+2}$

$a=3,\ b=2$

$c-d+1=0$

$2c+d-4=0$

$\frac{c}{4-1} = \frac{d}{2+4} = \frac{1}{1+2}$

$c=1,\ d=2$

$\therefore M = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

5) The eigen values of an idempotent matrix are i) all zero, ii) all 1, iii) 0 and 1, iv) none of these.

⟹ Let, A be an idempotent matrix.

$\therefore A^2 = A$

$\Rightarrow A^2 - A = 0$

$\therefore$ Characteristic eq^n is, $\lambda^2 - \lambda = 0$

$\Rightarrow \lambda(\lambda-1) = 0$

$\therefore \lambda = 0,\ \lambda = 1$

6) If all the characteristic roots of a matrix $A_{n\times n}$ is zero then show that A is a nilpotent matrix.

⟹ Here, the characteristic eq^n is, $\lambda^n = 0$

$\therefore$ By Cayley-Hamilton theorem, $A^n = 0$

$\therefore$ A is a nilpotent matrix.